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Effect of colored noise on networks of nonlinear oscillators

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We discuss noise-induced pattern formation in different two-dimensional networks of nonlinear oscillators, namely a sequence of biochemical reactions and the Lorenz system. The main focus of the work is on the dependence of these patterns on the correlation time (i.e., the color) of exponentially correlated Gaussian noise. It is seen that in the nonchaotic case, the homogeneity (or average cluster size) goes through a minimum with higher correlation time, while in its chaotic regime the Lorenz system shows a higher degree of synchronization when the correlation time of the noise is increased. In order to elucidate the origin of this phenomenon, the effect of colored noise on the individual oscillator is investigated. It is shown that the specific dependence of the network's homogeneity on the noise correlation time arises from an interplay of the collective behavior and the properties of the single oscillators.

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In models of biological systems, fluctuations are typically accounted for by white noise. However, for many systems, white noise is not an accurate approximation of the actual fluctuations present in the system. In these cases, colored noise may provide a more accurate description. As a rule, dynamics on a very large time scale (compared to the time scale of observation or the system's characteristic time θ) are ignored when formulating a mathematical model of the system, while those contributions to the overall dynamics with a time scale $t_N \ll \theta$ are thought of as noise [1]. An incomplete separation of such time scales into long ($t \gg \theta$), characteristic ($t \sim \theta$), and short ($t \ll \theta$) time constants, respectively, leads to a variety of additional dynamical effects. The existence of an intermediate regime between long and characteristic time scales may lead to dynamical bifurcations [2]. An intermediate regime between characteristic and short time scales can formally be accounted for by introducing correlations in the noise, i.e., by the use of colored noise. Similar to the case of dynamical bifurcations, one may expect a very different effect of such correlated noise on the observed dynamics in comparison to white noise. Recently, the influence of the noise color on stochastic resonance [3], on fluctuation-induced transport [4], on limit cycle and threshold behavior of individual and coupled oscillators [5–7], and on nonequilibrium phase transitions [8] has been discussed in several papers. For more details on the effect of noise color on nonlinear dynamical systems, see [9].

Here we present a new point of view in this discussion by studying the effect of colored noise on spatiotemporal cluster formation. Throughout the paper, we consider exponentially correlated Gaussian noise as a model for fluctuations. Within this model, the term "noise color" denotes the correlation time τ. It is seen that the homogeneity of patterns (or the average cluster size) in a system of coupled nonlinear differential equations strongly depends on the temporal correlation of additive, spatially incoherent noise. For chaotic networks, an increased correlation time τ increases the homogeneity of patterns, whereas in the nonchaotic case, a clear minimum of the homogeneity at intermediate τ is observed. In addition to being a novel phenomenon in the study of nonlinear networks under the influence of noise, this effect may be of relevance for applications in life sciences, as it suggests a possible mechanism of pattern regulation.

We discuss this effect in two different two-dimensional square lattices, whose elements consist of diffusively coupled, noisy nonlinear oscillators, namely a system of oscillatory biochemical reactions with backward inhibition, in the following called the Thron system (cf. [10] for the single oscillator and [11] for a square lattice of coupled oscillators),

$$\dot{x}_{ij} = \frac{a}{K + z_{ij}} - x_{ij},$$

$$\dot{y}_{ij} = x_{ij} - y_{ij},$$

$$\dot{z}_{ij} = y_{ij} - \frac{k_1 z_{ij}}{K_m + z_{ij}} + \xi_{ij}(t) + D_z \sum_{kl \in \mathcal{N}_{ij}} (z_{kl} - z_{ij}), \tag{1}$$

and the Lorenz system (cf. [12] for the single oscillator and [6] for a ring of coupled oscillators),

$$\dot{x}_{ij} = \alpha(y_{ij} - x_{ij}),$$

$$\dot{y}_{ij} = r x_{ij} - y_{ij} - x_{ij} z_{ij} + \xi_{ij}(t) + D_y \sum_{kl \in \mathcal{N}_{ij}} (y_{kl} - y_{ij}),$$

$$\dot{z}_{ij} = x_{ij} y_{ij} - b z_{ij}, \tag{2}$$

both with $1 \leq i,j \leq 32$. The neighborhood $\mathcal{N}_{ij}$ consists of the four nearest neighbors of each element within the square lattice. For our choice of parameters, each uncoupled oscillator in Eqs. (1) and (2) undergoes a Hopf bifurcation at $(a,K,K_m) = (0.1,0.001,0.001)$ for $k_1 = 0.46$ and $(\alpha,b) = (10,\frac{8}{3})$ at $r = 24.74$, respectively. In the following, we vary k_1 and r, keeping the other parameters fixed. In both cases, the individual oscillators are coupled diffusively in one of their dynamical variables with diffusion constants D_z for the Thron system and D_y for the Lorenz system. The quantity $\xi_{ij}(t)$ is spatially incoherent, exponentially correlated additive Gaussian noise, generated through an integral algorithm [13] using an Ornstein-Uhlenbeck process with Gaussian white noise of variance D. The resulting noise variable $\xi_{ij}(t)$ has zero mean and the correlation function obeys the equation $\langle \xi_{ij}(t)\xi_{ij}(t')\rangle = D\tau^{-1}\exp(-|t-t'|\tau^{-1})$, where τ is the correlation time and $D\tau^{-1}=\sigma^2$ is the variance of the noise. Spatial incoherence means that the noise is uncorrelated from site to site, i.e., $\langle \xi_{ij}(t)\xi_{i'j'}(t)\rangle = 0$. At this point, one basically has two possibilities to investigate the effect of noise color in these systems, namely at fixed σ^2 and at fixed D. While the latter leads to the correct white-noise limit, the former procedure, which has also been used in, e.g., [6], allows for approximately keeping the noise intensity constant (for finite τ). We believe that this case is the relevant situation for biological systems. In order to compare with other theoretical work, however, the case of constant D is more important.

Equations (1) and (2) were numerically integrated using a Heun algorithm [7] with a step size $\Delta t = 10^{-3}$ t.u. (time units). Periodic boundary conditions were applied for both networks. The choice of the diffusively coupled dynamical variable has been made mainly for the sake of comparison with previous work (i.e., Ref. [6] for the Lorenz system and Ref. [11] for the Thron system). However, we checked that a coupling in a different variable leads to the same effect.

We chose these two systems in order to give one example with explicit biochemical implications, and another that is somewhat standard in nonlinear dynamics in addition to having a strong physical background in hydrodynamics and laser physics. Additionally, we performed the same analysis for the Sel'kov model of glycolysis [14], for the Lorenz system both being in a regime of regular limit-cycle oscillations (at $r>312$), and for networks of driven double-well Duffing oscillators, of driven excitable oscillators [15], and of (chaotic) Rössler oscillators. For $\sigma^2=\text{const}$, they all follow the same scheme of response to noise color. On this basis, we are led to the conjecture that at fixed σ^2 the destructive effect with increasing noise color on nonlinear networks in their nonchaotic regime, as well as the constructive influence in the chaotic case, represent a widely observable, possibly universal phenomenon not restricted to some particular class of systems. Our conjecture is supported by the plausibility arguments for the individual oscillator given below. We believe that this effect is capable of providing insight into pattern-formation processes in biological systems: On the basis of the view towards noise presented in the introductory part of this paper, it can be argued that colored noise should be the rule rather than an exception in a biological system, where dynamical processes on all time scales contribute more or less directly to the observed dynamics (for a specific biological example, see [16]). By discussing explicitly the influence of colored noise on a network of biologically motivated oscillators such as the Thron system, we provide a framework for discussion of colored noise in life sciences.

The average synchronization between nearest neighbors is quantified using an inhomogeneity measure I [17], which is calculated in the following way:

$$I = \frac{1}{T}\int_0^T dt\left[\frac{1}{N}\sum_{ij}\sum_{kl\in\mathcal{N}_{ij}}(z_{kl}-z_{ij})^2\right], \tag{3}$$

where T is the length of the time series after the initial transients have died off, and N is the number of network elements. In addition to calculating the inhomogeneity, we determined the distribution of cluster sizes by applying the Hoshen-Kopelman algorithm for cluster classification [18]. The results of both procedures are in good agreement. Note that our means of quantification differ from Ref. [6], where the Euclidian distance in phase space has been used to quantify synchronization. Keeping in mind that we discuss aspects of spatiotemporal pattern formation, a quantification in just one dynamical variable seems more natural to us. We checked, however, that qualitatively the same results are obtained when one discusses synchronization, rather than inhomogeneity, as a function of the color. To calculate the noise-induced deviation from the deterministic limit-cycle trajectory for one oscillator, we introduce the following quantity:

$$\Delta = \left\langle \frac{\Delta t}{t_\Delta}\sum_{t=0}^{t_\Delta}\|\vec{u}_{\text{det}}(t)-\vec{u}_{\text{sto}}(t)\|\right\rangle_M, \tag{4}$$

where $\|\;\|$ represents the Euclidian distance between the deterministic ($\vec{u}_{\text{det}}$) and stochastic state vectors ($\vec{u}_{\text{sto}}$) at time t. The distance is zero at $t=0$. The brackets $\langle\;\rangle_M$ denote averaging over M different noise realizations for the fixed time interval t_Δ. Δt is the integration step size ($t_\Delta \gg \Delta t$). Varying t_Δ over a considerable range leads to the same qualitative results for the dependence on τ, even though the absolute values of Δ may differ substantially. Basically, Δ quantifies the area in phase space accessible for the system in the presence of noise at a given noise intensity and noise color. An increase of Δ with τ shows that a larger part of the phase space is mapped out by the trajectories within a fixed time interval t_Δ.

It is instructive to compare the τ dependence observed in the numerical simulations of the two systems, namely the regularly oscillating Thron system and the Lorenz system in its chaotic regime. To achieve a certain degree of comparability, the noise correlation time τ is measured in units of the dominating frequency ν_0 in the power spectrum of the individual oscillator. In the case of the Thron system, one has $\nu_0 = 0.095\ (\text{t.u.})^{-1}$, whereas for the Lorenz system one finds $\nu_0 = 1.30\ (\text{t.u.})^{-1}$.

For the Thron system, the results reported in Ref. [11] are fully reproduced. However, in contrast to their claim, our study of the spectrum of Lyapunov exponents for the coupled oscillators indicates that the time development is still regular, rather than chaotic. Indeed, calculation of the largest Lyapunov exponents of the deterministic networks revealed a periodic, nonchaotic behavior for the Thron network for any choice of D_z, $k_1 > 0.46$, and a chaotic behavior, as it should be, for the Lorenz network for $D_y < 0.16$ with $r>26$.

The central result of our investigation is given in Fig. 1. There the inhomogeneity I [Eq. (3)] is shown as a function of the noise correlation time τ. While the behavior is qualita-

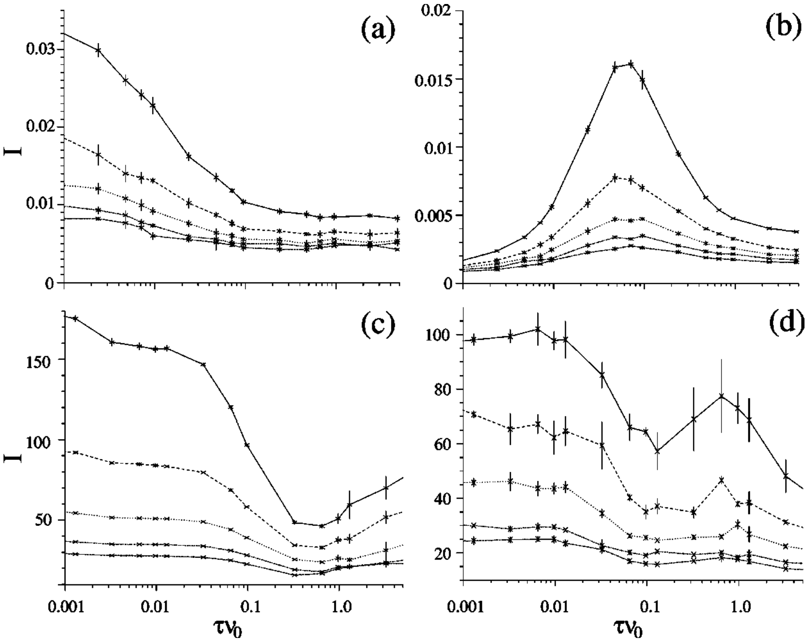

FIG. 1. The inhomogeneity I [Eq. (3)] as a function of $\tau\nu_0$ for different values of the coupling strength. D and σ^2 are held constant in the first and second column, respectively. (a,b) Thron system, $a=0.10$, $k_1=0.49$, $K=K_m=0.001$, $\nu_0=0.095$ $(\mathrm{t.u.})^{-1}$; D_z is varied from 0.6 (—) to 1.8 (-···-) in steps of 0.3. (c,d) Chaotic Lorenz system, $\alpha=10$, $b=\frac{8}{3}$, $r=28$, $\sigma^2=1.0$, $\nu_0=1.30$ $(\mathrm{t.u.})^{-1}$. D_y is varied from 0.15 (—) via 0.17 (- - -), 0.20 (· · ·), 0.25 (-·-), to 0.30 (-··-). (a) $D=0.02$, (b) $\sigma^2=0.02$, (c) $D=1.0$, (d) $\sigma^2=1.0$. The integration time was $T=2500$ t.u. with a transient of 100 t.u. Five realizations have been performed, each time using a different sequence of the attractor as well as different sets of random numbers for calculating the noise amplitudes. Error bars denote standard deviations.

tively the same at $D=\mathrm{const}$ [Figs. 1(a) and 1(c)], we see significant differences at $\sigma^2=\mathrm{const}$ [Figs. 1(b) and 1(d)]. For the Thron system, I shows a global maximum at some intermediate τ. In contrast, an overall decrease in a relevant regime of τ is observed for the chaotic Lorenz system. In both cases, the effect is diminished when the coupling is increased. For the Lorenz case, it is seen that an increase of the coupling constant from $D_y=0.15$ to 0.25 is sufficient to almost fully eliminate the dependence of the inhomogeneity on the color parameter τ. Our findings for the Lorenz system are in agreement with those discussed in [6] for the one-dimensional case.

In Fig. 2, typical snapshots of the two systems are shown for different values of τ. The effect of varying τ on the formation of patterns is clearly seen. The typical cluster size shows a clear minimum at intermediate τ for the Thron network [Figs. 1(a)–1(c)], whereas it increases with τ in the Lorenz case [Figs. 1(d)–1(f)].

Some insight into the systematics of the function $I(\tau)$ is gained when one compares the behavior seen in Fig. 1(d) with the corresponding curve for the Lorenz system while varying the bifurcation parameter r (Fig. 3). At values below the Hopf bifurcation $r=24.74$, one observes an increase of I for $\tau\nu_0<0.1$, similar to that of the Thron system. However, in the chaotic regime of the network $(r>26)$, this behavior changes drastically, and one regains the function $I(\tau)$ from Fig. 1(d), with precisely the opposite global properties to that of the Thron system. In order to understand the above phenomena, it is worthwhile to look at the behavior of the individual oscillator under the influence of colored noise. It has been shown in [19] that the effect of noise on limit-cycle oscillations depends on the phase of the oscillation.

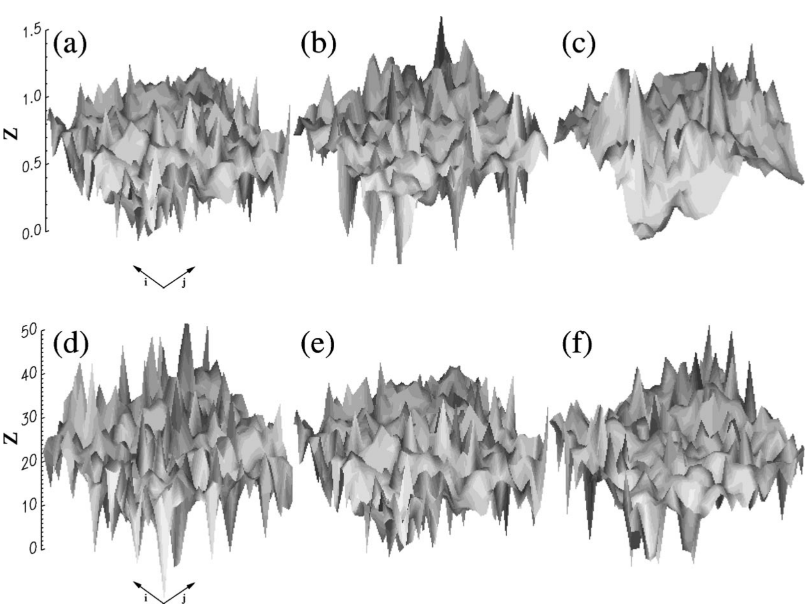

FIG. 2. Snapshots of the networks of 32×32 coupled Thron and Lorenz oscillators at different values of τ keeping $\sigma^2=\mathrm{const}$. Shown are the z variables. (a)–(c) The Thron system at $\tau\nu_0=0.01,0.1,1$, $\sigma^2=0.02$, $D_z=0.6$. (d)–(f) The Lorenz system at $\tau\nu_0=0.01,0.1,5$, $\sigma^2=1.0$, $D_y=0.15$. Other parameters are the same as in Fig. 1.

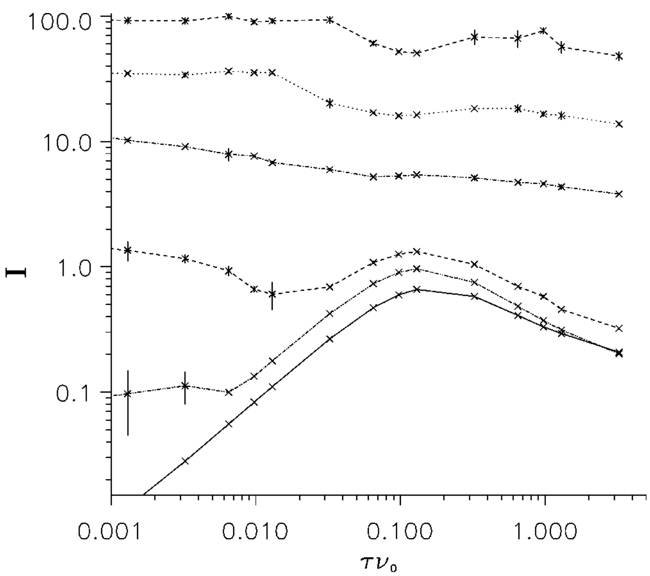

FIG. 3. The inhomogeneity I as a function of $\tau\nu_0$ for the Lorenz system at different values of r, while keeping σ^2 constant. From top to bottom: $r=28$, $r=27$, $r=26$, $r=25$, $r=24.5$, and $r=20$; $D_y=0.15$. Other parameters are the same as in Fig. 1(d).

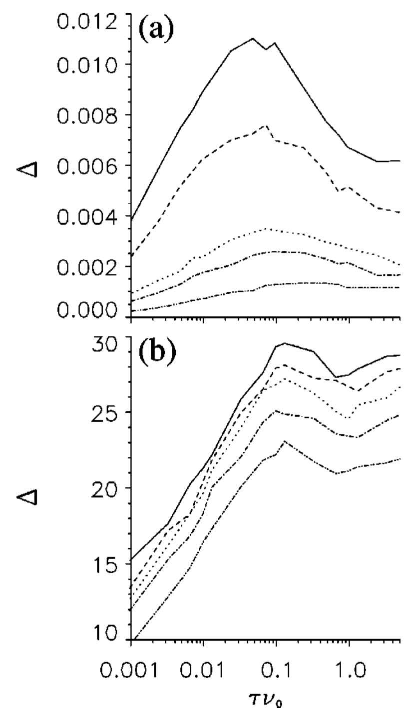

FIG. 4. The average distance Δ [Eq. (4)] between the perturbed and unperturbed phase-space trajectories of the individual oscillator as a function of $\tau\nu_0$ for different values of σ^2. (a) Thron system. σ^2 is varied from 0.01 (-$\cdot\cdot$-) via 0.05 (-$\cdot$-), 0.1 ($\cdots$), 0.5 (- - -), to 1.0 (—), $t_\Delta=10.27$ t.u., $M=10^4$. (b) Chaotic Lorenz system. σ^2 is varied from 0.5 (-$\cdot\cdot$-) via 1.0 (-$\cdot$-), 1.5 ($\cdots$), 2.0 (- - -), to 2.5 (—), $t_\Delta=5.0$ t.u., $M=10^4$. Other oscillator parameters are the same as in Fig. 1.

To quantify the perturbation of a trajectory as a function of σ^2 and τ, the average distance Δ [Eq. (4)] between the unperturbed trajectory in phase space and the trajectory under the influence of noise is applied. Figures 4(a) and 4(b) show Δ as a function of the correlation time τ for the Thron system and the chaotic Lorenz system, respectively.

Quite remarkably, on the level of the individual oscillator, no fundamental difference between the chaotic Lorenz system and the oscillatory Thron system is observed. The difference between the curves in Figs. 1(c) and 1(d) thus has to be related either to a fully collective effect or to the interplay between the collective and the single-oscillator response to colored noise. For the case of the chaotic Lorenz oscillator, this interplay can partially be understood when one compares the τ dependence of the average distance Δ from Fig. 4 with the function $I(\tau)$ from Fig. 1. On the one hand, the diffusive coupling immediately exploits any (noise-induced) freedom in the choice of the phase-plane trajectory, to synchronize with the neighboring oscillators. This is a collective effect, as it involves nearest-neighbor considerations and, clearly, it reduces the inhomogeneity. While for sufficiently large noise intensity this effect basically can be found for any value of the color parameter τ, owing to the properties of the *single* oscillator shown in Fig. 4, this mechanism of synchronization is more relevant at higher τ, resulting in a decrease of the inhomogeneity I as a function of τ. This collective effect is responsible for the global tendency of $I(\tau)$ to display an overall decrease. On the other hand, the contribution of the *individual* oscillator tends to increase the inhomogeneity of the network, as its phase-space trajectory is less localized at higher τ leading to the observed local maximum around $\tau\nu_0=1.0$.

For the Thron network, the position of the maximum in Fig. 1(b) could be estimated by performing a similar analysis to the one in [19]. Within a single oscillation, a small time interval Δs exists, when the system's trajectory is perturbed most severely by noise. For colored noise, this results in a time scale matching between τ and Δs. The position of the maximum is given to a good approximation by $\tau\Delta s^{-1}\approx 1$.

Concluding, in the present paper it has been shown how patterns in different two-dimensional networks of nonlinear oscillators depend on the correlation time of external noise sources. All systems we investigated follow a characteristic scheme of response to colored noise: chaotic oscillations lead to a decrease, regular oscillations lead to a maximum of the pattern's inhomogeneity at intermediate τ. As soon as one accepts that a biological system itself may serve as the source of noise, e.g., via processes at faster time scales than the characteristic time of the studied system, colored noise has to be regarded as a frequent phenomenon in biological systems. It is by now well established that noise is functionally exploited in nature (see, e.g., [20]). In a similar way, the regulation of patterns by noise color, as reported here, could have a role in biological systems. Following this line of thought, one is led to a new view towards aspects of pattern formation (e.g., animal coating on the basis of a biochemical switch) and developmental biology.

Several open questions remain. It has to be studied in more detail how coupling strength and noise amplitude influence this behavior. In addition, it would be worthwhile to systematically discuss the same effect in a network with an external periodic stimulus. Previous work indicates that the noise correlation parameter in such a driven network has a strong effect on the network's synchronization [21].

Furthermore, the phenomenon reported here may provide a method for distinguishing spatiotemporal chaos from (high-dimensional) noise. When probed by external colored noise, a system displaying chaotic spatiotemporal dynamics should give a very different response from that for a system with a regular behavior masked by an additional (internal) noise. This approach, addressing the analysis of experimental data rather than the conceptual understanding of pattern formation, needs further theoretical investigation.

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